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Stability of a viscous liquid curtain

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The stability of a viscous liquid curtain falling down steadily under the influence of gravity is investigated. Both spatially and temporally changing disturbances are considered in the linear analysis. Only the spatially growing sinuous disturbances whose group velocity points toward upstream are unstable. The group velocity is in the upstream direction only when the Weber number of the curtain flow exceeds $\frac{1}{2}$. The predicted critical Weber number agrees completely with that found experimentally by Brown (1961). The viscosity is shown to have the dual roles of increasing the amplification rate as well as the damping rate of the disturbances.

1. Introduction

The dynamics of thin sheets of liquids has been studied extensively by G. I. Taylor (1959*a*, *b*, *c*). The subject is of considerable scientific and technological importance. Clark & Dombrowski (1972), Crapper *et al.* (1973), Crapper, Dombrowski & Pyott (1975) and Weihs (1978) have recently studied the disintegration of liquid sheets in connexion with atomization, combustion and spray coatings. Brown (1961) studied experimentally the general behaviour of a liquid sheet in the context of curtain coating.

A thin sheet of viscous liquid flowing between two vertical guide wires is an integral part of a process called curtain coating. It also has the potential of being used as a device for dynamic surface tension measurements. Some description of this process and its applications in various industries can be found in the references of Brown's work. Brown found that his measured velocity distribution in the curtain compares closely with the prediction based on a nonlinear differential equation attributed to G. I. Taylor. He observed that the curtain will disintegrate if the flow rate is reduced to a certain minimum value. He also discussed curtain stability on the basis of a simple momentum balance applied to a stationary free edge resulting from the curtain breaking. However, the effect of viscosity and the dynamics of instability are not considered, and the mechanism at work in curtain instability remains unclear. The purpose of this study is to fill in this information gap with the help of linear stability analysis.

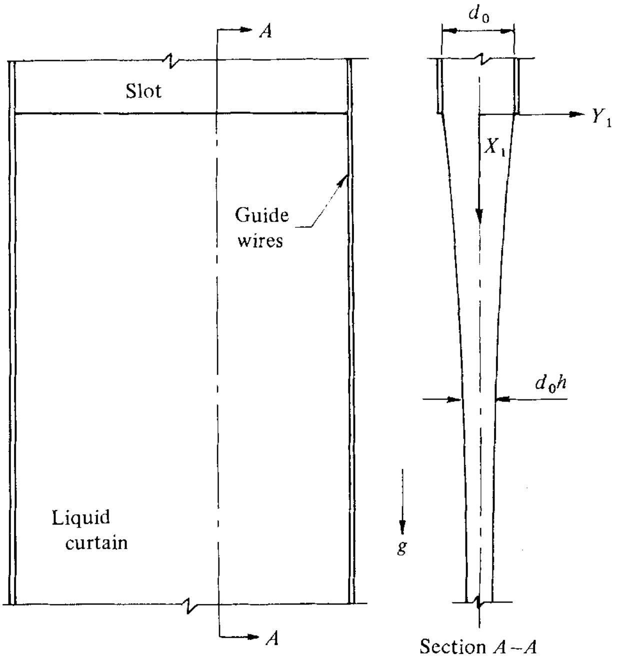

FIGURE 1. Definition sketch.

2. Stability analysis

2.1. *Basic flow*

Consider the steady flow in the Newtonian liquid curtain shown in figure 1. According to Brown (1961), Taylor derived the following nonlinear differential equation for the velocity distribution in the curtain.

$$\left(\frac{1}{U}U_X\right)_X+\frac{1}{U}-U_X=0,\quad U_X+V_Y=0, \tag{1}$$

where the subscripts denote differentiations, and U and V are respectively the dimensionless velocity components in the directions of the Cartesian co-ordinates (X, Y). These dimensionless quantities are related to their dimensional counterparts (U_1, V_1) and (X_1, Y_1) by

$$(U_1, V_1)=(4\mu g/\rho)^{\frac{1}{3}}(U, V), (X_1, Y_1)=(4\mu/\rho)^{\frac{2}{3}}g^{-\frac{1}{3}}(X, Y),$$

where μ is the dynamic viscosity, ρ the density, and the distance X_1 is measured from the upper edge of the curtain in the direction of the gravitational acceleration g. Brown found good agreements between his measured velocities and the numerical results obtained from (1) by Maruo (1958: Maruo's results were cited by Brown 1961 as private communication through G. I. Taylor). It should be pointed out that (1) was derived with the assumptions that the flow is essentially two dimensional and that the effects of surface tension as well as the normal stress variation across the curtain are negligible to the first-order approximation. These assumptions are borne out by Brown's experiments for sufficiently thin curtains.

2.2. *Stability equations*

The stability of the described basic flow with respect to two dimensional disturbances are to be investigated. Substituting the perturbed flow quantities into the Navier–Stokes equation, we arrive at the dimensionless equations

$$\left.\begin{aligned} &u_t+(\bar{u}+u)\,u_x+u\bar{u}_x+(\bar{v}+v)\,u_y+v\bar{u}_y = -p_x+(u_{xx}+u_{yy})/R,\\ &v_t+(\bar{u}+u)\,v_x+u\bar{v}_x+(\bar{v}+v)\,v_y+v\bar{v}_y = -p_y+(v_{xx}+v_{yy})/R,\\ &u_x+v_y = 0, \end{aligned}\right\} \tag{2}$$

where all subscripts denote partial differentials, (x, y) are the Cartesian co-ordinates in the unit of the maximum curtain thickness d_0, $(\bar{u}, \bar{v})$ and (u, v) are respectively the (x, y) components of the primary flow velocity and the velocity perturbations in the unit of $\bar{u}_0 = Q/d_0$, Q being the volumetric flow rate per unit width of the curtain, t is time measured in the unit of $d_0/\bar{u}_0$, p the pressure perturbation non-dimensionalized by $\rho\bar{u}_0^2$, and R is the Reynolds number defined by

$$R = \rho\bar{u}_0 d_0/\mu.$$

In this study we consider only the case of a gradually varying curtain thickness. It is easily verified that

$$\bar{u}_x = (R/4F^2)^{\frac{1}{3}}\,U_X, \quad F = \bar{u}_0^2/gd_0 \equiv \text{Froude number.}$$

According to figure 5 of Brown, $U_X = O(1)$. Thus, $\bar{u}_x = \delta U_X = -\bar{v}_y \ll 1$ if

$$(R/4F^2)^{\frac{1}{3}} = (g^2/4\nu)^{\frac{1}{3}}\,(d_0^2/Q) = \delta \ll 1.$$

For the case of thin curtains such that $\delta \ll 1$, we define a slow variable ξ and apply the method of multiple scale to write

$$\xi = \delta x, \quad \partial_x \to \partial_x + \delta\partial_\xi.$$

By use of the above relations and neglecting terms of $O(\delta)$ as well as the nonlinear terms in perturbations, we reduce the first two equations in (2) to

$$\left.\begin{aligned} u_t+\bar{u}(\xi)\,u_x &= -p_x+(u_{xx}+u_{yy})/R,\\ v_t+\bar{u}(\xi)\,v_x &= -p_y+(v_{xx}+v_{yy})/R. \end{aligned}\right\} \tag{3}$$

The third equation is automatically satisfied by the stream function ψ related to the velocity perturbations by

$$u = \psi_y, \quad v = -\psi_x. \tag{4}$$

Upon elimination of the pressure terms by cross differentiations, (3) can be written in terms of ψ as:

$$\left[\partial_t+\bar{u}\,\partial_x-\frac{1}{R}(\partial_{xx}+\partial_{yy})\right](\partial_{xx}+\partial_{yy})\,\psi = 0. \tag{5}$$

Equation (5) is the governing differential equation of the linear stability problem under consideration.

Let the free surfaces of the basic flow and the perturbed flow be respectively

$$y = \pm\frac{h}{2}(x) \quad \text{and} \quad y = \pm\frac{h}{2}(x)+\eta(x,t) = \zeta(x,t).$$

The kinematic condition at the free surface $y = \zeta$ requires that

$$v = \zeta_t + (\bar{u}+u)\zeta_x. \tag{6}$$

The dynamic condition of the free surface, which is massless by definition, demands that the net force be zero at the free surface. Demanding the vanishing of the force per unit area of the free surface in the x and y directions, we have respectively

$$\left[-p+\frac{2}{R}(\bar{u}+u)_x\right]\zeta_x - [(\bar{u}+u)_y+(\bar{v}+v)_x]/R \mp WK\zeta_x = 0 \tag{7}$$

and

$$\left[-p+\frac{2}{R}(\bar{v}+v)_y\right] - [(\bar{u}+u)_y+(\bar{v}+v)_x]\zeta_x/R \mp WK = 0, \tag{8}$$

where K is the total surface curvature and W is the Weber number,

$$K = \frac{(\pm\frac{1}{2}h+\eta)_{xx}}{[1+(\pm\frac{1}{2}h+\eta)_x^2]^{\frac{3}{2}}}, \quad W = \frac{T}{\rho\bar{u}_0^2 d_0};$$

T is the surface tension.

Note that $h_x = O(\delta)$ since $Q = (\bar{u}_0\bar{u})(d_0 h) =$ constant and $\bar{u}_x = O(\delta)$. Neglecting terms of $O(\delta)$, balancing out purely primary flow quantities, and expanding the remaining primary flow quantities in Taylor's series about $y = \pm\frac{1}{2}h$, and then retaining only linear terms, we reduce the above boundary conditions at $y = \zeta$ to the following to be applied at $y = \pm\frac{1}{2}h$:

$$\eta_t + \bar{u}\eta_x + \psi_x = 0, \tag{9}$$

$$\psi_{yy} - \psi_{xx} = 0, \tag{10}$$

$$\pm W\eta_{xx} + p + 2\psi_{xy}/R = 0, \tag{11}$$

where p can be obtained from (3) in terms of ψ.

Equation (5) and its boundary conditions (9), (10) and (11) constitute a linear eigenvalue problem. We consider for our solution a normal mode of travelling disturbances

$$\psi = \phi(y)\exp[i\alpha(x-ct)], \tag{12}$$

where $\alpha = 2\pi d_0/\lambda$, λ is the wavelength, and c is the wave speed in the unit of $\bar{u}_0$. We consider both cases of temporally and spatially growing disturbances. For the formal case α is real but $c = c_R + ic_I$ is complex. For the latter case $\alpha = \alpha_R + i\alpha_I$ is the complex wavenumber but $\alpha c = \omega$ is the real wave frequency. Thus, temporally changing disturbances are stable or unstable depending on if $c_I < 0$ or $c_I > 0$, and spatially changing disturbances are stable or unstable depending on if $\alpha_I > 0$ or $\alpha_I < 0$.

Substituting (12) into (5), we have

$$[i\alpha R(c-\bar{u}) + (d^2-\alpha^2)](d^2-\alpha^2)\phi = 0, \quad d^2 = d^2/dy^2.$$

The general solution of this equation is

$$\phi = A\sinh(\alpha y) + B\cosh(\alpha y) + C\sinh(My) + D\cosh(My)$$

where A, B, C and D are integration 'constants' depending on ξ, and

$$M^2 = \alpha^2 - i\alpha Rc', \quad c' = c - \bar{u}(\xi).$$

Since the governing differential system is linear and homogeneous, we may consider the even and odd solutions for ϕ separately.

2.3. *Varicose waves*

The odd solution for ϕ corresponds to the anti-symmetric disturbances which displace each of the free surfaces in opposite directions. Substituting

$$\phi = A \sinh(\alpha y) + C \sinh(My)$$

into (4) and (3) and solve for p, we have

$$p = \alpha c' A \cosh(\alpha y) \exp[i\alpha(x - ct)].$$

The solution of (9) for η gives

$$\eta = \pm c'^{-1}[A \sinh(\tfrac{1}{2}\alpha h) + C \sinh(\tfrac{1}{2}Mh)] \exp[i\alpha(x - ct)].$$

Substitution of the expressions for p and η and (12) into (10) and (11) yields

$$A[2\alpha^2 \sinh(\tfrac{1}{2}\alpha h)] + C[(M^2 + \alpha^2) \sinh(\tfrac{1}{2}Mh)] = 0,$$

$$A[c' \cosh(\tfrac{1}{2}\alpha h) - Wc'^{-1}\alpha \sinh(\tfrac{1}{2}\alpha h) + 2R^{-1}i\alpha \cosh(\tfrac{1}{2}\alpha h)] + C[2R^{-1}iM \cosh(\tfrac{1}{2}Mh) - W\alpha c'^{-1} \sinh(\tfrac{1}{2}Mh)] = 0.$$

The existence of nontrivial solutions for A and C requires

$$c'^2 + c'(2i\alpha/R)[1 - 2\alpha M(M^2 + \alpha^2)^{-1} \coth(\tfrac{1}{2}Mh) \tanh(\tfrac{1}{2}\alpha h)] + W\alpha \tanh(\tfrac{1}{2}\alpha h)[2\alpha^2/(M^2 + \alpha^2) - 1] = 0. \quad (13)$$

For both temporally and spatially growing disturbances of long wave-lengths, $\alpha \to 0$ near the neutral stability curve and the secular equation (13) can be expanded in powers of α as

$$(c - \bar{u})^2 + i4R^{-1}\alpha(c - \bar{u}) - \tfrac{1}{2}Wh\alpha^2 + O(\alpha^3) = 0. \quad (14)$$

For the temporal case α is real and c is complex. The solution of (14) for c gives

$$c - \bar{u} = -2i\alpha/R \pm \alpha[\tfrac{1}{2}Wh - (2/R)^2]^{\frac{1}{2}}.$$

Thus

$$c_I = -2\alpha/R, \quad c_R = \bar{u} \pm \alpha[\tfrac{1}{2}Wh - (2/R)^2]^{\frac{1}{2}} \quad \text{if} \quad \tfrac{1}{2}Wh - (2/R)^2 > 0,$$

and

$$c_I = -2\alpha/R \pm [(2/R)^2 - \tfrac{1}{2}Wh]^{\frac{1}{2}}\alpha, \quad c_R = \bar{u} \quad \text{if} \quad \tfrac{1}{2}Wh - (2/R)^2 < 0.$$

It follows that $c_I < 0$ regardless of whether the wave speed relative to the fluid particle is zero or not. Therefore, the temporally changing varicose disturbances are damped with a dimensional damping rate given by

$$\frac{2}{R}\alpha^2 \frac{\bar{u}_0}{d_0} = \frac{8\pi^2\nu}{\lambda^2}. \quad (15)$$

To investigate the spatially growing disturbances of long wave-lengths, we multiply (14) by α^2 and identify αc with ω to have

$$(\omega - \alpha\bar{u})^2 + i4R^{-1}(\omega - \alpha\bar{u})\alpha^2 - \tfrac{1}{2}Wh\alpha^4 + O(\alpha^5) = 0.$$

The solution of this equation in powers of small ω gives the following complex wave-number

$$\alpha = \frac{\omega}{\bar{u}} + \frac{2i}{R}\frac{\omega^2}{\bar{u}^3} + O(\omega^3).$$

i.e.

$$\alpha_R = \frac{\omega}{\bar{u}} = \frac{\alpha_R c}{\bar{u}}, \quad \alpha_I = \frac{2}{R}\frac{\omega^2}{\bar{u}^3} = \frac{2\alpha_R^2}{R\bar{u}} > 0.$$

Thus the spatially varying disturbances are also damped travelling waves.

It is easily verified that the obtained spatial amplification factor α_I and the temporal growth rate c_I are related, as they should be, by the theorem of Gaster (1962)

$$\alpha_R c_I = -\alpha_I \frac{\partial}{\partial \alpha_R}(\alpha_R c_R). \tag{16}$$

Numerical solutions of (13) for finite values of α have not yielded undamped disturbances for either the temporal or spatial case.

It has been shown that the liquid curtain is stable with respect to both spatially and temporally varying disturbances of the varicose mode. However, there is another mode of disturbances associated with the even solution of ϕ. This mode of disturbance displaces both free surface of the curtain in the same direction to form sinuous waves. It will be shown presently that, while the curtain is also stable with respect to temporally varying disturbances of the sinuous mode, it may become unstable owing to the spatially growing sinuous disturbances.

2.4. *Sinuous waves*

The analysis required to yield the secular equation for the sinuous mode is identical to that for the varicose mode, except that the hyperbolic sine and tangent functions must be replaced, respectively, by the hyperbolic cosine and cotangent functions and vice versa. Thus the secular equation for the sinuous mode can be put down at once, inferring directly from (13),

$$c'^2 + c'(2i\alpha/R)\left[1 - 2\alpha M(M^2+\alpha^2)^{-1}\tanh\left(\tfrac{1}{2}Mh\right)\coth\left(\tfrac{1}{2}\alpha h\right)\right] + W\alpha\coth\left(\tfrac{1}{2}\alpha h\right)\left[2\alpha^2/(M^2+\alpha^2) - 1\right] = 0. \tag{17}$$

Consider first the temporally varying disturbances of long wave lengths such that $\alpha \to 0$. Expanding the above equation in powers of α, we have

$$(c-\bar{u})^2\left[1 + \tfrac{1}{3}\alpha^2h^2 + O(\alpha^4)\right] + i(c-\bar{u})\left[h^2\alpha^3/3R + O(\alpha^5)\right] - W\left[2/h + \tfrac{1}{6}h\alpha^2 + O(\alpha^4)\right] = 0.$$

The solution of this equation for the complex wave speed gives

$$\left.\begin{aligned} c_R &= \bar{u} \pm (2W/h)^{\frac{1}{2}}\left[1 - \tfrac{1}{8}(\alpha h)^2 + O(\alpha^4)\right], \\ c_I &= -\frac{h^2}{6R}\alpha^3\left[1 + \frac{\alpha^2h^2}{3} + O(\alpha^4)\right]^{-1}. \end{aligned}\right\} \tag{18}$$

Thus the temporally varying disturbances of long wavelengths are weakly dispersive and damped. The dimensional damping rate is

$$\frac{h^2}{6R}\alpha^4\frac{\bar{u}_0}{d_0} = \frac{\nu d_0^2}{6}\left(\frac{2\pi}{\lambda}\right)^4 h^2. \tag{19}$$

Comparing (15) with (19), we find that while the damping rates of varicose and sinuous modes of temporally varying disturbances are both linearly proportional to the kinematic viscosity, the former is independent of d_0 but the latter is directly proportional to d_0^2. Moreover, the varicose disturbances are damped more rapidly than the sinuous ones in the same curtain.

The amplification rate of spatially growing disturbances can be obtained by use of Gaster's equation (16),

$$\alpha_I = \tfrac{1}{6}\alpha_R^4 R^{-1}h^2[\bar{u} \pm (2W/h)^{\frac{1}{2}}\{1-\tfrac{3}{8}(\alpha_R h)^2\}]^{-1}$$
$$= \tfrac{1}{6}\alpha_R^4 R^{-1}h^2/(\text{group velocity of disturbances}).$$

The same results can be obtained directly as was done for the case of varicose waves, but will not be demonstrated here. Note that $\alpha_I > 0$ or $\alpha_I < 0$ depending on if the disturbance group velocity is positive or negative. Thus the spatially growing disturbances whose group velocity is in the downstream direction is stable but those whose group propagates upstream is unstable. It is seen from the expression of α_I that viscosity plays the role of stabilizing agent when the curtain is stable but turns around and acts as a destabilizing agent when the curtain is unstable. The neutral curve corresponding to zero group velocity is given by

$$\bar{u}-(2W/h)^{\frac{1}{2}}[1-\tfrac{3}{8}(\alpha_R h)^2] = 0.$$

By use of the relation $\bar{u}h = 1$, this equation can be written as

$$1-(2Wh)^{\frac{1}{2}}[1-\tfrac{3}{8}(\alpha_R h)^2] = 0. \tag{20}$$

Thus the neutral curve for the long wave disturbances is a parabola in the α_R, W plane. Numerical solutions of (17) for finite α_R confirm the asymptotic relation (20), and show that the neutral stability curve is monotonic. Thus, the critical weber number W_c occurs at $\alpha = 0$, and is given by

$$W_c = [(2h)^{-1}]_{\min} = \tfrac{1}{2}. \tag{21}$$

Note the curtain thickness decreases in the direction of flow and thus $h(\xi) = \bar{d}/d_0 < 1$, $\bar{d}$ being the dimensional curtain thickness. A liquid curtain is unstable if $W > W_c$, since then $\alpha_I < 0$. This prediction agrees completely with Brown's (1961) experimental observations that the viscous liquid curtain is unstable if

$$2T/\rho Q\bar{u}\bar{u}_0 > 1.$$

3. Conclusion

A curtain of viscous liquid flowing between two guide wires such that

$$\delta = (g^2/4\nu)^{\frac{1}{3}}(d_0^2/Q) \ll 1$$

is shown to be stable with respect to temporally as well as spatially changing varicose disturbances. The curtain is also stable with respect to temporally varying sinuous disturbances but may become unstable with respect to spatially growing sinuous disturbances. The latter disturbances experience either an exponential growth or decay depending on if their group velocity is opposite to or in the direction of the basic flow. The group velocity is in the upstream direction only when the Weber number exceeds $\frac{1}{2}$. The predicted critical Weber number agrees completely with that experimentally found by Brown. It is implicit in Gaster's equation (16), that if the group velocity is opposite to the basic flow direction, ($-x$ direction), then the temporal and the spatial formulations of the linear stability problem will lead to opposite conclusions. The present analysis for the case of sinuous disturbances offer an excellent example. The same equation also implies that both temporal and spatial formulations will lead to the same conclusion on the stability criteria if the group velocity is in the direction of the flow. Our analysis for varicose disturbances is a good example. Another example is furnished by the stability problem of a liquid layer flow down an inclined plane. The equivalence of the temporal and spatial formulations for this problem has been discussed recently by Lin (1975) and Krantz (1975).

It should be pointed out that we neglected terms of $O(\delta)$ in the analysis. Thus our results based on the quasi-parallel flow approximation are valid first order solutions which predict the first order effects in a liquid curtain with sufficient accuracy only if its thickness is so thin that $\delta \ll 1$. For such a liquid curtain the omitted higher order terms in the linear analysis may well be less significant than the neglected nonlinear effects. It will be of interest to know if the conclusions on stability reached by the present normal mode analysis will be altered by the findings from the corresponding stability analysis in the frame work of initial value problems. Although more general disturbances which can be constructed from normal modes by Fourier superposition will give the same stability criteria, the disturbance corresponding to any continuous spectrum in the initial value problem may not. However, based on the good agreement between our theory and known experiments, we conjecture that the transient part of the solution to the initial value problem will be damped and the normal mode solution recovered.

It is seen from (18) that the speed of sinuous disturbances decreases as the curtain thickness increases. Therefore the disturbances which propagate upstream will experience overturning when they are forced to overtake the waves in front of them, since the curtain thickness increases in the upstream direction. It is very unlikely that one will find supercritical stability in the nonlinear analysis. However, there is an evidence of sub-critical instability. Brown observed that if the disturbance amplitude is so large as to cause the two free surfaces to meet the curtain will break, even if $W < W_c$, to form an inverted V-shaped free edge. G. I. Taylor (1959*b*) in fact demonstrated that $W_c = \frac{1}{2}$ from a momentum balance for an element of such a free edge in a broken sheet of an inviscid liquid.

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REFERENCES

Brown, D. R. 1961 *J. Fluid Mech.* **10**, 297.
Clark, C. J. & Dombrowski, N. 1972 *Proc. Roy. Soc.* A **329**, 467.
Crapper, G. D., Dombrowski, N., Jepson, W. P. & Pyott, G. A. D. 1973 *J. Fluid Mech.* **57**, 671.
Crapper, G. D., Dombrowski, N. & Pyott, G. A. D. 1975 *Proc. Roy. Soc.* A **342**, 209.
Gaster, M. 1962 *J. Fluid Mech.* **14**, 222.
Krantz, W. B. 1975 *A.I.Ch.E. J.* **21**, 179.
Lin, S. P. 1975 *A.I.Ch.E. J.* **21**, 178.
Taylor, G. I. 1959*a* *Proc. Roy. Soc.* A **253**, 289.
Taylor, G. I. 1959*b* *Proc. Roy. Soc.* A **253**, 296.
Taylor, G. I. 1959*c* *Proc. Roy. Soc.* A **253**, 313.
Weihs, D. 1978 *J. Fluid Mech.* **87**, 289.